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Spectral properties of entanglement witnesses

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Abstract

Entanglement witnesses are observables which when measured, detect entanglement in a measured composed system. It is shown what kind of relations between eigenvectors of an observable should be fulfilled, to allow an observable to be an entanglement witness. Some restrictions on the signature of entanglement witnesses, based on an algebraic-geometrical theorem will be given. The set of entanglement witnesses is linearly isomorphic to the set of maps between matrix algebras which are positive, but not completely positive. A translation of the results to the language of positive maps is also given. The properties of entanglement witnesses and positive maps are expressed as special cases of general theorems for k -Schmidt witnesses and k -positive maps. The results are therefore presented in a general framework.

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1. Introduction

The crucial resource in quantum-informational science is quantum entanglement. It provides an essential difference between quantum and classical information theory. It is therefore of special importance, to know the techniques of measuring and quantifying entanglement. Despite enormous efforts in the last decade, and a number of significant partial results, the problem how to determine that a given density matrix possesses this desired resource or not, is not solved in general. One of the most important methods of detecting entanglement exploits a class of special observables called *entanglement witnesses*. In this paper, the entanglement witnesses are considered in the language of their spectral decomposition.

Only finite-level quantum systems, which are composed of two subsystems, will be considered. The first one is a d_1 -level system, and the second one is a d_2 -level system. We assume, that $d_1 \leq d_2$. The Hilbert space of the system is then a tensor product of the Hilbert spaces of its subsystems: $\mathcal{H} = \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$. By fixing the orthogonal basis $\{e_i\}_{i=1}^{d_1}$ as the basis of \mathbb{C}^{d_1} , and the orthogonal basis $\{f_j\}_{j=1}^{d_2}$ as the basis of \mathbb{C}^{d_2} , one gets the linear isomorphism

$$\mathfrak{A} : \mathcal{H} \rightarrow \mathbb{M}_{\mathbb{C}}(d_1 \times d_2)$$

between vectors in $\mathcal{H} = \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ and $d_1 \times d_2$ complex matrices of their coordinates, defined by the formula

$$[\mathfrak{A}(\Psi)]_{i,j} = \langle e_i \otimes f_j | \Psi \rangle. \tag{1}$$

The vector Ψ can be therefore expressed by the matrix $\mathfrak{A}(\Psi)$ as

$$\Psi = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} [\mathfrak{A}(\Psi)]_{ij} |i\rangle |j\rangle.$$

Now the *Schmidt rank* of a vector Ψ in \mathcal{H} is defined as the rank of its coordinate matrix $\mathfrak{A}(\Psi)$. Denote the set of vectors of the Schmidt rank non-greater than k as \mathcal{S}_k . We then have the ascending sequence of closures of orbits of the group $GL(d_1) \times GL(d_2)$

$$\mathcal{S}_1 \subsetneq \dots \subsetneq \mathcal{S}_{d_1} = \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}.$$

Any matrix $A \in \mathbb{M}_{\mathbb{C}}(d_1 \times d_2)$ of rank one can be written as $v^T u$, where v and u are vectors respectively in \mathbb{C}^{d_1} and \mathbb{C}^{d_2} . Taking $a \in \mathbb{C}^*$ one gets the new pair $u' = a \cdot u, v' = v/a$ representing the same vector. Aliasing such pairs, we get the isomorphism $\mathbb{C}^{d_1} \times \mathbb{C}^{d_2} / \mathbb{C} \cong \mathcal{S}_1$.

Next define a projection

$$\mathfrak{P} : |\Psi\rangle \rightarrow \frac{|\Psi\rangle\langle\Psi|}{\langle\Psi|\Psi\rangle}$$

from $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ to the projective space $\mathbb{C}P^{d_1 \times d_2 - 1}$. The projective space $\mathbb{C}P^{d_1 \times d_2 - 1} = \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} / \mathbb{C}$ is the space of rays in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, i.e. the space $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \setminus \{0\}$ where the vectors of the same direction (equal up to multiplication by a nonzero scalar) are aliased (for a detailed discussion about the projective geometry see [1, 2]). The space of rays of the Hilbert space of a given quantum system is in one-to-one correspondence with the set of pure quantum states of the system.

Now projecting the above sequence of inclusions, one gets the following new ascending sequence of sets of projectors onto one-dimensional subspaces

$$\begin{array}{ccccccc} \mathcal{S}_1 & \subsetneq & \dots & \subsetneq & \mathcal{S}_{d_1} & = & \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \\ \downarrow \mathfrak{P} & & & & \downarrow \mathfrak{P} & & \downarrow \mathfrak{P} \\ \mathbb{C}P^{d_1-1} \times \mathbb{C}P^{d_2-1} & \cong & \mathfrak{P}(\mathcal{S}_1) & \subsetneq & \dots & \subsetneq & \mathfrak{P}(\mathcal{S}_{d_1}) = \mathbb{C}P^{d_1 \times d_2 - 1}, \end{array}$$

where the isomorphism in the lower sequence constitutes the Segre embedding $\mathbb{C}P^{d_1-1} \times \mathbb{C}P^{d_2-1} \hookrightarrow \mathbb{C}P^{d_1 \times d_2 - 1}$ (see [1, 2] for details). The elements of the set $\mathfrak{P}(\mathcal{S}_1)$ are called *pure separable states*. We define *mixed separable states* in accord with the work of Werner [3] as convex combinations of pure separable states.

Following Terhal *et al* [5], we extend this definition to higher k and define *density operators of Schmidt number k* [5, 6].

Definition 1. *The state ρ is called to be of the Schmidt number k , when it can be decomposed as a convex combination of projectors*

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|,$$

where the Schmidt ranks of the vectors ψ_i is less or equal to k , and cannot be decomposed as a convex sum of projectors onto vectors of Schmidt rank less than k .

The set of all states of Schmidt number less or equal to k is convex—it is the convex roof of the set $\mathfrak{P}(\mathcal{S}_k)$. The problem of separability is then a problem of membership in a convex set, with given extremal points. One of the ways to handle this problem is by using the

concept of *entanglement witnesses* introduced in [4]. Taking any entangled state ρ , we have two compact, convex, non-empty subsets of the linear coset of operators of rank 1—the set of separable states and the singleton of the chosen entangled state. Now according to Banach separation theorem, there exists an affine subspace \hat{V} of codimension 1 (in the considered coset), which separates these two sets. We can extend now this subspace uniquely to a linear subspace V of codimension 1 in the Hilbert space of Hermitian operators. Now using the self-duality of this Hilbert space, one can assign to the subspace V of codimension 1 the unique (up to multiplication by nonzero scalar) observable W , such that V is its orthogonal complement. Such an observable is called an *entanglement witness*. We easily extend this definition to k -Schmidt witnesses (see [6]).

Definition 2. *The k -Schmidt witness is an observable which fulfils the conditions:*

- $\forall \Psi \in \mathcal{S}_k \langle \Psi | W | \Psi \rangle \geq 0$ (all states of Schmidt number less or equal to k are on the same side of V).
- for some ρ $\langle \rho | W \rangle_{HS} = \text{Tr}(\rho W) < 0$ (the singleton of ρ lies on the other side of V).

The problem of classification of k -Schmidt witnesses remains unsolved in general. In low dimensions (2×2 , 2×3) we have such a classification. Any entanglement witness (1-Schmidt witness) is of the form

$$W = A^\Gamma + B, \quad A, B \geq 0,$$

where Γ denotes the partial transposition in one of the subsystems (see [12]). Such witnesses are called *decomposable*, and states which can be detected by witnesses from this class are called NPT (*negative partial transposition*) entangled states. In higher dimensions this class of entanglement witnesses are a proper subset of the set of all witnesses. Entangled states not detected by this class are called PPT (*positive partial transposition*) entangled states. We have a simple criterion to check, whether a given state is NPT. The most interesting are then non-decomposable witnesses and tools to detect PPT entanglement based on them.

2. The main theorems

Having a Hermitian observable W , we define a decomposition of its domain due to the spectral decomposition of W

$$\mathbb{C}^{d_1} \times \mathbb{C}^{d_2} = V_+ \oplus V_- \oplus V_0$$

where the *positive subspace* V_+ is spanned by eigenvectors corresponding to positive eigenvalues, the *negative subspace* V_- is spanned by eigenvectors corresponding to negative eigenvalues, and V_0 is a kernel of W . Next due to the spectral decomposition one can represent W as a difference of two positive operators $W = W_+ - W_-$, the first one supported on V_+ , and the second one on V_- . Their kernels are $\ker W_\pm = V_0 \oplus V_\mp$.

Having a vector Ψ given by its Schmidt decomposition $\Psi = \sum_i \lambda_i \alpha_i \otimes \beta_i$, we define a subspace

$$\tilde{V}_\Psi = \text{span}\{\text{span}\{\alpha_i\} \otimes \mathbb{C}^{d_2} \cup \mathbb{C}^{d_1} \otimes \text{span}\{\beta_i\}\}.$$

It is also possible to define this subspace independently of the basis as

$$\tilde{V}_\Psi = \text{span}\{\text{Im Tr}_2 |\Psi\rangle\langle \Psi| \otimes \mathbb{C}^{d_2} \cup \mathbb{C}^{d_1} \otimes \text{Im Tr}_1 |\Psi\rangle\langle \Psi|\}.$$

where Im denotes the image or range of the operator. Partial traces Tr_1 and Tr_2 are defined as follows: $\text{Tr}_1 \rho = \sum_{ik} \delta_{ik} \rho_{ij,kl}$, respectively $\text{Tr}_2 \rho = \sum_{jl} \delta_{jl} \rho_{ij,kl}$, where $\rho_{ij,kl}$ denotes the matrix element of ρ in standard basis, i.e. $\langle e_i \otimes f_j | \rho | e_k \otimes f_l \rangle$.

Theorem 1. *Let W be a k -Schmidt witness. Its eigenvectors satisfy the three following conditions:*

- (i) $V_- \supsetneq \{0\}$.
- (ii) $\forall \Psi \in \mathcal{S}_k, \Psi \in V_0 \oplus V_- \Rightarrow \Psi \in V_0$.
- (iii) $\forall \Psi \in \mathcal{S}_k, \cap V_0 \tilde{V}_\Psi \cap (V_- \oplus V_0) \subset V_0$.

Proof.

- (i) A k -entanglement witness should detect something, so there are some states, for which the mean value of the witness on them is negative (the second condition in definition 2).
- (ii) If $\Psi \in \mathcal{S}_k \cap V_0 \oplus V_-$, then Ψ is a combination of eigenvectors only from the kernel and V_- . Due to the first condition in definition 2, the mean value of the observable on the vector Ψ should be nonnegative, so Ψ can be only a combination of the eigenvectors from the kernel.
- (iii) To prove the necessity of the third condition, observe that when an observable W is a k -SW, then $\forall \Psi \in \mathcal{S}_k \langle \Psi | W_+ | \Psi \rangle - \langle \Psi | W_- | \Psi \rangle \geq 0$. This implies the condition that the supremum of the function

$$F : \mathcal{S}_k \setminus V_0 \oplus V_- \rightarrow \mathbb{R} \quad F(\Psi) = \frac{\langle \Psi | W_- | \Psi \rangle}{\langle \Psi | W_+ | \Psi \rangle}$$

is less than 1. The function F plays a key role in this proof.

Let us take a vector $\sum_{i=1}^k x_i \otimes y_i \in \mathcal{S}_k$ in V_0 and some other vector $\sum_{i=1}^k \tilde{x}_i \otimes \tilde{y}_i \in \mathcal{S}_k$. Consider now the family of vectors from \mathcal{S}_k

$$\Phi(t) = \sum_{i=1}^k (x_i + t\tilde{x}_i) \otimes (y_i + t\tilde{y}_i). \tag{2}$$

This family forms an algebraic curve (over \mathbb{R}) of degree 2. The intersection $\{\Phi(t) : t \in \mathbb{R}\} \cap V_0$ can be one of the following sets:

- (i) The whole curve— $\{\Phi(t) : t \in \mathbb{R}\} \subset V_0$.
- (ii) $\{\Phi(t) : t \in \mathbb{R}\}$ meets the kernel in not more than two points. Because $\sum_{i=1}^k x_i \otimes y_i \in V_0$, one of these points is 0.

To begin with, consider the second case. In this case the root of the denominator of F_Φ in 0 is separated, i.e.

$$\exists \epsilon : \{\Phi(t) : t \in (-\epsilon, \epsilon)\} \cap V_0 = \{\Phi(0)\} = \sum_{i=1}^k x_i \otimes y_i \in V_0.$$

The restriction of the function F to the set $\Phi((-\epsilon, \epsilon) \setminus \{0\})$ gives a function $F_\Phi : (-\epsilon, \epsilon) \setminus \{0\} \rightarrow \mathbb{R}$, given by the formula

$$F_\Phi(t) = \frac{\langle \Phi(t) | W_- | \Phi(t) \rangle}{\langle \Phi(t) | W_+ | \Phi(t) \rangle}. \tag{3}$$

Computing the numerator of F_Φ and taking into account that $W_\pm | \Phi \rangle = 0$ one gets

$$\begin{aligned} \langle \Phi(t) | W_- | \Phi(t) \rangle &= \left\langle \sum_{i=1}^k (x_i + t\tilde{x}_i) \otimes (y_i + t\tilde{y}_i) \middle| W_- \middle| \sum_{i=1}^k (x_i + t\tilde{x}_i) \otimes (y_i + t\tilde{y}_i) \right\rangle \\ &= t^2 \left\langle \sum_{i=1}^k \tilde{x}_i \otimes \tilde{y}_i \middle| W_- \middle| \sum_{i=1}^k \tilde{x}_i \otimes \tilde{y}_i \right\rangle \end{aligned}$$

$$\begin{aligned}
 & + 2t \operatorname{Re} \left\langle \sum_{i=1}^k x_i \otimes \tilde{y}_i + \sum_{i=1}^k \tilde{x}_i \otimes y_i \middle| W_- \middle| \sum_{i=1}^k \tilde{x}_i \otimes \tilde{y}_i \right\rangle \\
 & + \left\langle \sum_{i=1}^k x_i \otimes \tilde{y}_i + \sum_{i=1}^k \tilde{x}_i \otimes y_i \middle| W_- \middle| \sum_{i=1}^k x_i \otimes \tilde{y}_i + \sum_{i=1}^k \tilde{x}_i \otimes y_i \right\rangle. \tag{4}
 \end{aligned}$$

For the denominator we get

$$\begin{aligned}
 \langle \Phi(t) | W_+ | \Phi(t) \rangle & = \left\langle \sum_{i=1}^k (x_i + t\tilde{x}_i) \otimes (y_i + t\tilde{y}_i) \middle| W_+ \middle| \sum_{i=1}^k (x_i + t\tilde{x}_i) \otimes (y_i + t\tilde{y}_i) \right\rangle \\
 & = t^2 \left(t^2 \left\langle \sum_{i=1}^k \tilde{x}_i \otimes \tilde{y}_i \middle| W_+ \middle| \sum_{i=1}^k \tilde{x}_i \otimes \tilde{y}_i \right\rangle \right. \\
 & \quad + 2t \operatorname{Re} \left\langle \sum_{i=1}^k x_i \otimes \tilde{y}_i + \sum_{i=1}^k \tilde{x}_i \otimes y_i \middle| W_+ \middle| \sum_{i=1}^k \tilde{x}_i \otimes \tilde{y}_i \right\rangle \\
 & \quad \left. + \left\langle \sum_{i=1}^k x_i \otimes \tilde{y}_i + \sum_{i=1}^k \tilde{x}_i \otimes y_i \middle| W_+ \middle| \sum_{i=1}^k x_i \otimes \tilde{y}_i + \sum_{i=1}^k \tilde{x}_i \otimes y_i \right\rangle \right). \tag{5}
 \end{aligned}$$

For a given $\{\tilde{x}_i, \tilde{y}_i\}$, we get a rational function $F_\Phi : \mathbb{R} \rightarrow \mathbb{R}$. Such a function is the quotient of two quadratic polynomials

$$F_{\{\tilde{x}_i, \tilde{y}_i\}}(t) = \frac{at^2 + bt + c}{dt^2 + et + f}. \tag{6}$$

Now the assumption that the supremum of F on its domain is finite, implies that for any curve Φ , the limit of F_Φ in 0 is finite.

First consider the degenerated subcase, when $\sum_{i=1}^k \tilde{x}_i \otimes \tilde{y}_i \in V_0$. Then $a = d = b = e = 0$, and f is nonzero (because we assume at the moment, that the curve Φ is not a subset of V_0). The limit of F_Φ in 0 is finite.

We can then consider the generic case, when $d \neq 0$. At the beginning observe, that $f = 0 \Rightarrow e = 0, c = 0 \Rightarrow b = 0$. Suppose that the denominator has a root in 0 (by the above observation, it is then a double root), which means that $f = 0 \iff \sum_{i=1}^k x_i \otimes \tilde{y}_i + \sum_{i=1}^k \tilde{x}_i \otimes y_i \in \ker W_+$ (by formula (5) and the positivity of W_+). Finiteness of the limit of F_Φ in 0 (according to formula (4) and the positivity of W_-) now implies that also $c = 0 \iff \sum_{i=1}^k x_i \otimes \tilde{y}_i + \sum_{i=1}^k \tilde{x}_i \otimes y_i \in \ker W_-$ (then also $b = 0$ and the numerator has also a double root in 0). The limit is then equal to a/d , which is finite. Using the definition of the subspace $\tilde{V}_{\sum x_i \otimes y_i}$ and considering that $\ker W_+ \cap \ker W_- = V_0$, we can rewrite this implication in the following form:

$$\forall \phi \in \tilde{V}_{\sum x_i \otimes y_i} \phi \in V_0 \oplus V_- \Rightarrow \phi \in V_0, \tag{7}$$

what gives the assertion of the theorem.

It remains to consider the situation, when $\{\Phi(t) : t \in \mathbb{R}\} \subset \ker W_+$. This implies that

$$\begin{aligned}
 \sum_{i=1}^k \tilde{x} \otimes \tilde{y} & \in \ker W_+ \\
 \sum_{i=1}^k (\tilde{x} \otimes y + x \otimes \tilde{y}) & \in \ker W_+.
 \end{aligned}$$

But now by the second point of the theorem, also $\{\Phi(t) : t \in \mathbb{R}\} \subset \ker W_-$, so in particular

$$\sum_{i=1}^k (\tilde{x} \otimes y + x \otimes \tilde{y}) \in \ker W_-.$$

Again, identifying the kernels and using the definition of $\tilde{V}_{\sum x_i \otimes y_i}$ we get the assertion of the theorem. \square

Under some stronger assumptions about the subspaces V_0, V_+ and V_- , one can prove the opposite theorem:

Theorem 2. Consider the decomposition of $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ into direct sum of mutual orthogonal subspaces: $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} = V_+ \oplus V_0 \oplus V_-$. Let W_{\pm} be arbitrary positive operators supported on V_{\pm} .

Define $\hat{V} = \bigcap_{\Psi \in \mathcal{S}_k \cap V_0} \tilde{V}_{\Psi}^{\perp}$. If

- (i) $V_- \supsetneq \{0\}$,
- (ii) $\forall \Psi \in \mathcal{S}_k, \Psi \in V_0 \oplus V_- \Rightarrow \Psi \in V_0$,
- (iii) $V_- \subset \hat{V}$,

then for large enough λ the observable $\lambda W_+ - W_-$ is an k -SW.

Proof. (i) The first condition guarantees the existence of detected states, so the first condition in definition (2) is fulfilled.

One can decompose any k -separable vector Ψ as

$$\Psi = \hat{\Psi} + \tilde{\Psi},$$

where $\hat{\Psi} \in \hat{V}$ and $\tilde{\Psi} \in \hat{V}^{\perp}$. Now using the decomposition $W = W_+ - W_-$ we can write the second condition in definition (2) as

$$\begin{aligned} \forall \hat{\Psi} \in \hat{V} \forall \tilde{\Psi} \in \hat{V}^{\perp} : \hat{\Psi} + \tilde{\Psi} \in \mathcal{S}_k \langle \hat{\Psi} + \tilde{\Psi} | W_+ | \hat{\Psi} + \tilde{\Psi} \rangle \\ \geq \langle \hat{\Psi} + \tilde{\Psi} | W_- | \hat{\Psi} + \tilde{\Psi} \rangle = \langle \hat{\Psi} | W_- | \hat{\Psi} \rangle. \end{aligned} \tag{8}$$

The last equality holds because of the third assumption. When $\hat{\Psi} = 0$, then condition (8) is fulfilled. It is then enough to focus on vectors Ψ for which $\hat{\Psi} \neq 0$. When condition (8) is fulfilled for a vector Ψ , then also for a vector $\alpha \Psi$, where $\alpha \in \mathbb{C}^*$. It is then sufficient to restrict the quantificated set to the set of vectors Ψ , for which $\|\hat{\Psi}\| = 1$. Condition (8) now takes the form

$$\begin{aligned} \forall \hat{\Psi} \in \hat{V} : \|\hat{\Psi}\| = 1 \forall \tilde{\Psi} \in \hat{V}^{\perp} : \hat{\Psi} + \tilde{\Psi} \in \mathcal{S}_k \\ \langle \hat{\Psi} + \tilde{\Psi} | W_+ | \hat{\Psi} + \tilde{\Psi} \rangle \geq \langle \hat{\Psi} | W_- | \hat{\Psi} \rangle. \end{aligned} \tag{9}$$

\hat{V} is a product space. To see it, let us introduce the notations

$$\begin{aligned} A_{\Psi} &= \text{Im Tr}_2 |\Psi\rangle\langle\Psi| \\ B_{\Psi} &= \text{Im Tr}_1 |\Psi\rangle\langle\Psi| \end{aligned}$$

and write the subspace \tilde{V}_{Ψ} using them

$$\tilde{V}_{\Psi} = \text{span}\{A_{\Psi} \otimes \mathbb{C}^{d_2} \cup \mathbb{C}^{d_1} \otimes B_{\Psi}\}.$$

Its orthogonal complement is then equal

$$\begin{aligned} \tilde{V}_{\Psi}^{\perp} &= (A_{\Psi} \otimes \mathbb{C}^{d_2})^{\perp} \cap (\mathbb{C}^{d_1} \otimes B_{\Psi})^{\perp} \\ &= A_{\Psi}^{\perp} \otimes \mathbb{C}^{d_2} \cap \mathbb{C}^{d_1} \otimes B_{\Psi}^{\perp} = A_{\Psi}^{\perp} \otimes B_{\Psi}^{\perp}. \end{aligned}$$

The subspace \hat{V} is an intersection of such subspaces, so it is also a product subspace

$$\begin{aligned} \hat{V} &= \bigcap_{\Psi \in \mathcal{S}_k \cap V_0} \tilde{V}_\Psi^\perp = \bigcap_{\Psi \in \mathcal{S}_k \cap V_0} (A_\Psi^\perp \otimes B_\Psi^\perp) \\ &= \left(\bigcap_{\Psi \in \mathcal{S}_k \cap V_0} A_\Psi^\perp \right) \otimes \left(\bigcap_{\Psi \in \mathcal{S}_k \cap V_0} B_\Psi^\perp \right). \end{aligned}$$

Denote $\hat{V}_1 = \bigcap_{\Psi \in \mathcal{S}_k \cap V_0} A_\Psi^\perp$ and $\hat{V}_2 = \bigcap_{\Psi \in \mathcal{S}_k \cap V_0} B_\Psi^\perp$.

Because $\hat{V} = \hat{V}_1 \otimes \hat{V}_2$ is a product subspace, after the appropriate change of basis of \mathbb{C}^{d_1} and \mathbb{C}^{d_2} the matrix $\mathfrak{A}(\hat{\Psi})$ of an element $\hat{\Psi} \in \hat{V}$ is a matrix, which has zeros out of some subblock. To see it, set such basis $\{e\}_{i=1}^{d_1}, \{f\}_{j=1}^{d_2}$ of \mathbb{C}^{d_1} and \mathbb{C}^{d_2} respectively, that $\hat{V}_1 = \text{span}\{e_1, \dots, e_{\dim \hat{V}_1}\}$ and $\hat{V}_2 = \text{span}\{f_1, \dots, f_{\dim \hat{V}_2}\}$. Now the matrix $\mathfrak{A}(\hat{\Psi})$ of any element $\hat{\Psi} \in \hat{V}$ has zeros outside the left-upper corner of the size $\dim \hat{V}_1 \times \dim \hat{V}_2$.

From now, we use such basis. The matrix $\mathfrak{A}(\hat{\Psi})$ is therefore nonzero only in its left-upper subblock of size $\dim \hat{V}_1 \times \dim \hat{V}_2$, and the matrix $\mathfrak{A}(\tilde{\Psi})$ is therefore nonzero only outside its left-upper subblock of size $\dim \hat{V}_1 \times \dim \hat{V}_2$. This means that a given vector $\Psi = \hat{\Psi} + \tilde{\Psi}$ is k -separable, only if the vector $\hat{\Psi}$ is k -separable. (If a matrix has the rank lower than k , then any its subblock has the rank lower than k .) Now for a given normalized vector $\hat{\Psi} \in \mathcal{S}_k \cap \hat{V}$ define the set

$$U_{\hat{\Psi}} = \{\tilde{\Psi} \in \hat{V}^\perp : \hat{\Psi} + \tilde{\Psi} \in \mathcal{S}_k\}. \tag{10}$$

Using this observation, one can rewrite condition (9) in the following form:

$$\begin{aligned} \forall \hat{\Psi} \in \hat{V} \cap \mathcal{S}_k : \|\hat{\Psi}\| = 1 \forall \tilde{\Psi} \in U_{\hat{\Psi}} \\ \langle \hat{\Psi} + \tilde{\Psi} | W_+ | \hat{\Psi} + \tilde{\Psi} \rangle \geq \langle \hat{\Psi} | W_- | \hat{\Psi} \rangle. \end{aligned} \tag{11}$$

The sets $U_{\hat{\Psi}}$ are:

- Non-empty: zero is always an element.
- Closed (in the metric topology of \hat{V}^\perp): by the isomorphism \mathfrak{A} defined by (1), the set $\mathfrak{A}(U_{\hat{\Psi}})$ is a set of matrices having their left-upper block of the size $\dim \hat{V}_1 \times \dim \hat{V}_2$ fixed and the rank less or equal to k (direct reformulation of definition (10)). It is a set of common zeros of all minors of rank $k + 1$ (the matrix entries from left-upper block we treat as fixed constants). The minors are polynomial functions, and hence continuous, so the set $U_{\hat{\Psi}}$ is the inverse image of a closed set $\{0\}$, and therefore closed.

Now define a continuous function $f_{\hat{\Psi}} : U_{\hat{\Psi}} \rightarrow \mathbb{R} \cup \{0\}$ by the left side of the inequality (9)

$$f_{\hat{\Psi}} = \langle \hat{\Psi} + \tilde{\Psi} | W_+ | \hat{\Psi} + \tilde{\Psi} \rangle. \tag{12}$$

By the second and the third assumption, we have

$$\forall \Psi \in \mathcal{S}_k \quad \Psi \in V_0 \oplus V_- \Rightarrow \Psi \in V_0 \Rightarrow \Pi_{\hat{V}} \Psi = 0$$

(here $\Pi_{\hat{V}}$ denotes an orthogonal projection). Now, by the contraposition rule

$$\forall \Psi \in \mathcal{S}_k \quad \Pi_{\hat{V}} \Psi \neq 0 \Rightarrow \Psi \notin V_0 \oplus V_- \Rightarrow \langle \Psi | W_+ | \Psi \rangle > 0.$$

Therefore on any $\tilde{\Psi} \in U_{\hat{\Psi}}$ the function $f_{\hat{\Psi}}$ takes strictly non-negative values. The image of the domain of $f_{\hat{\Psi}}$ is then a closed subset of \mathbb{R}_+ . The infimum of the function $f_{\hat{\Psi}}$ on its domain is then strictly positive. Denote it by $c_{\hat{\Psi}}$.

Consider now the continuous function $c : \hat{V} \rightarrow c_{\hat{\Psi}}$, defined on the set $\{\hat{\Psi} : \hat{\Psi} \in \hat{V} \wedge \|\hat{\Psi}\| = 1\}$. The domain is compact, so the function c is bounded from below, the lower

bound C is reached at some point of the domain and then is positive. This result allows us to bound the left-hand side of the inequality in (9) from below as

$$\langle \hat{\Psi} + \tilde{\Psi} | W_+ | \hat{\Psi} + \tilde{\Psi} \rangle \geq C.$$

On the same set $\{\hat{\Psi} : \hat{\Psi} \in \hat{V} \cap \mathcal{S}_k \wedge \|\hat{\Psi}\| = 1\}$ we have also given a real-valued continuous function by the right-hand side of the inequality (9)

$$g(\hat{\Psi}) = \langle \hat{\Psi} | W_- | \hat{\Psi} \rangle.$$

Again, because of the compactness of the domain, this function is bounded from above. Denote its maximum by G .

Now by rescaling the positive part W_+ one obtains $C \geq G$. This ensures, that the inequality in (8) is fulfilled for any vector of the Schmidt rank k , and then the observable is a k -SW. \square

The next theorem will allow us to give some restrictions on the signature of k -Schmidt witnesses when the dimensions of the subsystems are given.

Lemma 1. *The set $\mathcal{S}_k \subset \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ is an affine variety of dimension $d_1 \times d_2 - (d_1 - k) \times (d_2 - k)$.*

Proof. By the isomorphisms \mathfrak{A} defined by (1) we can treat the set \mathcal{S}_k as a set of $d_1 \times d_2$ matrices of rank k . It is an affine variety generated by the ideal of all minors of rank $k + 1$. There are $\binom{d_1}{k+1} \times \binom{d_2}{k+1}$ such minors, but locally only $(d_1 - k) \times (d_2 - k)$ of them are independent.

For any regular point of this variety A (a matrix of the rank equal to k) we can find basis in \mathbb{C}^{d_1} and \mathbb{C}^{d_2} , respectively $\{f\}_{i=1}^{d_1}$ and $\{e\}_{i=1}^{d_2}$ such that $A = \sum_{i=1}^k f_i^T e_i$. On this basis a minor built from the first k columns and the first k rows is nonzero. There exists an open neighbourhood of this regular point (in the space $\mathbb{M}_{\mathbb{C}}(d_1 \times d_2)$), such that any matrix in this neighbourhood has its first k columns and first k rows linearly independent. To check whether a given matrix from this neighbourhood has rank less or equal to k , one has to check $(d_1 - k) \times (d_2 - k)$ independent conditions

$$\det \left[\begin{array}{ccc|c} b_{11} & \dots & b_{1k} & b_{1i} \\ \vdots & \ddots & \vdots & \vdots \\ b_{k1} & \dots & b_{kk} & b_{ki} \\ \hline b_{j1} & \dots & b_{jk} & b_{ji} \end{array} \right] = 0,$$

(in the new basis). Dimension of the tangent space in a regular point, and hence the dimension of the variety is then equal to $d_1 \times d_2 - (d_1 - k) \times (d_2 - k)$. \square

For detailed discussion and proofs of the facts about manifolds of matrices of a fixed rank see [7] or [8].

Theorem 3. *Any subspace $V \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ of dimension $\dim V > (d_1 - k) \times (d_2 - k)$ contains a k -separable vector.*

Proof. Because the variety \mathcal{S}_k is defined by a zero of an ideal of homogenous polynomials, one can consider its projectivization of the dimension $d_1 \times d_2 - (d_1 - k) \times (d_2 - k) - 1$. It is a subvariety of $\mathbb{C}P^{d_1 \times d_2 - 1}$. Consider now another subvariety of $\mathbb{C}P^{d_1 \times d_2 - 1}$ —a projectivization of linear subspace $V \subset \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$. Its dimension is $\dim V - 1$. Projective dimension theorem [2] says that if the sum of dimensions of two subvarieties of $\mathbb{C}P^N$ is greater or equal

to N , then their intersection is a nonempty set. Using this for the above subvarieties, we find that if

$$d_1 \times d_2 - (d_1 - k) \times (d_2 - k) - 1 + \dim V - 1 \geq d_1 \times d_2 - 1,$$

then subspace V consists of some k -separable vectors. □

The theorem used is a generalization of the well-known fact, that two projective lines on a projective plane always have an intersection. To show that there is no strict restriction on this dimension, I give an example of basis (non-orthogonal), which spans the $(d_1 - k) \times (d_2 - k)$ -dimensional subspace, which does not contain any k -separable vector.

Theorem 4. *The set $\{\sum_{i=1}^k e_{m+i} \otimes f_{n+i}\}_{m \leq d_1 - k, n \leq d_2 - k}$ spans the subspace V_{\max}^k which does not contain any nonzero k -separable vector.*

Proof. Again using the isomorphism \mathfrak{A} defined by (1), we will prove that any nonzero matrix in the subspace $V_{\max}^k \subset \mathbb{M}_{\mathbb{C}}(d_1 \times d_2)$ spanned by the set of matrices $\{\sum_{i=1}^k e_{m+i}^T f_{n+i}\}_{m \leq d_1 - k, n \leq d_2 - k}$ has rank greater than k .

We call any subset of matrix elements with a constant difference between indices a *diagonal* of a matrix (in general rectangular). This difference will be called the *number* of a diagonal. The number of a diagonal varies between $1 - d_1$ and $d_2 - 1$.

The proof will be carried out by induction with respect to the number of a diagonal. Any matrix in V_{\max}^k has the first k and the last k of diagonals equal to zero (any diagonal which has less than k elements is equal to zero). Now assume, that there exists a matrix of the rank k in V_{\max}^k .

We will prove that if all diagonals of numbers less than p are zero, then also the p th diagonal is zero.

Any minor of rank $k + 1$ is equal to zero, in particular the minors whose diagonal (in standard sense) is created from elements of the considered p th diagonal. Because we assume that all diagonals with their numbers less than p are equal to zero, such minors are determinants of upper-triangular matrices, and therefore the products of $k + 1$ elements from the p th diagonal. If all such minors are equal to zero, then all products of $k + 1$ elements from the considered diagonal are equal to zero. All basis elements which give a nonzero contribution to the p th diagonal are $v_m = \sum_{i=1}^k e_{m+i} \otimes f_{n+i}$, where $n - m = p$. In the same way let us number the elements of diagonal b_m . The relation between the elements of diagonal and the coefficients of combination of v_m is given by the following system of linear equations:

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & & \ddots & 0 \\ 0 & \ddots & & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{k+1} \\ \vdots \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_{k+1} \\ \vdots \end{bmatrix}. \tag{13}$$

Because any product of $k + 1$ elements from the considered diagonal is equal to zero, at most k equations can have the right-hand side different from 0. Removing any k rows from the matrix in (13), we have a nonsingular square matrix. Because we have already removed all equations with nonzero right-hand side, the system (13) has only zero solution. By induction, we then zero all diagonals, so the only matrix in V_{\max}^k with rank less than k is the matrix with all coefficients equal to 0. □

3. Propositions and examples

Proposition 1. *Any eigenvector of a k -SW related to a negative eigenvalue is not k -separable.*

Proof. Use the second condition in theorem 1. □

Theorem 3 leads to proposition:

Proposition 2. *For k -SW, $1 \leq \dim V_- \leq (d_1 - k) \times (d_2 - k)$.*

If in V_0 there are no k -separable vectors, then the third condition in theorem 1 is fulfilled, and then we have the following:

Proposition 3. *If $\ker W \cap \mathcal{S}_k = \emptyset$ and W is an k -SW, then*

$$\dim V_+ \geq k(d_1 + d_2) - k^2.$$

Proof. By the second condition in theorem 1, if there is a k -separable vector in $V_0 \oplus V_-$ then it belongs to V_0 . Because by the assumption no k -separable vectors are in V_0 , no k -separable vectors are in $V_0 \oplus V_-$, and therefore the dimension of $V_0 \oplus V_-$ is bounded from above by $(d_1 - k) \times (d_2 - k)$. Because $\dim V_+ = d_1 d_2 - \dim(V_- \oplus V_0)$, we get the above inequality. □

It is not possible to establish any non-trivial upper bound for positive subspace of an arbitrary k -Schmidt witness, because we can always get a new k -SW by adding a positive observable, with its support contained in the kernel of the witness. An interesting question would be, whether there exist such bounds for optimal witnesses, i.e. such witnesses, from which no positive observable cannot be subtracted without destroying the property of being k -Schmidt witness.

We can construct in practice examples of subspaces, which do not contain any separable vector, by means of *unextendible product basis* (denoted further as UPB, see [9, 10]), as the orthogonal complement of a subspace spanned by UPB. Such a subspace by the definition of UPB contains no separable vectors. The maximal number n of vectors in UPB in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ is bounded from below by the number

$$n \geq \sum_i (d_i - 1) + 1$$

(see lemma 1 in [9]). Subtracting it from the dimension of the space, we get an upper bound for the dimension of the constructed subspace, which does not contain any separable vectors

$$\dim V \leq (d_1 - 1) \times (d_2 - 1).$$

Then, for subspaces constructed by means of UPB, proposition 2 recovers the previous results.

Now for a subspace V with no non-negative separable vectors, theorem 2 assures us that an observable

$$W = I - (1 + \epsilon)P_V$$

is an EW for small enough positive ϵ . (P_V denotes a projection onto subspace V). For subspaces constructed by means of UPB, such a construction was made in [11]. Moreover, the nondecomposability of such W was proven.

Example 1. Any EW for two qubits has exactly one negative eigenvalue and exactly three positive eigenvalues.

Proof. At least one negative eigenvalue is needed, to ensure that the witness detects anything (first condition in definition 2). The eigenvector related to this eigenvalue is of Schmidt rank two. It will be denoted as Ψ^- . The negative space cannot consist of any separable vector, so proposition 2 bounds its dimension by 1.

No vector of Schmidt rank greater than one can be in the kernel. If there were such a vector Ψ , then by Nullstellensatz there exists such $\alpha : \beta$, that

$$\text{rank}(\alpha \mathfrak{A}(\Psi^-) + \beta \mathfrak{A}(\Psi)) = 1,$$

but then $\beta \neq 0$, so there would be a separable vector $\alpha\Psi + \beta\Psi^- \in V_0 \oplus V_-$, but $\alpha\Psi + \beta\Psi^- \in V_0 \oplus V_- \notin V_0$, which would be in contradiction with the second condition in theorem 1. Thus there can be only separable vectors in the kernel.

Write such a vector in its Schmidt decomposition

$$\mathfrak{A}(\Psi^0) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The vector Ψ^- is of Schmidt rank greater than one and orthogonal to Ψ^0 and can be written as

$$\mathfrak{A}(\Psi^-) = \begin{bmatrix} 0 & a \\ b & c \end{bmatrix},$$

where $ab \neq 0$. Now let us identify all separable vectors in $V_0 \oplus V_-$

$$\text{rank}(\alpha \mathfrak{A}(\Psi^-) + \beta \mathfrak{A}(\Psi^0)) = 1 \iff \beta(\alpha c + \beta ab) = 0.$$

If $c \neq 0$, then there exists a separable vector, which is in $V_0 \oplus V_-$, but not in V_0 , which is in contradiction with the second condition in theorem 1. Thus $c = 0$.

If there is any other vector in the kernel, let us say Ψ_1^0 , then by the condition that no entangled vector is in V_0 , the subspace V_0 consists of only separable vectors. We then conclude that $\mathfrak{A}(\Psi_1^0)$ has the second column or the second row equal to 0, but no such vector (other than Ψ^0) is orthogonal to Ψ^- . Therefore there is at most one vector in the kernel (up to scaling by nonzero scalar) and it is separable.

Let us identify now the subspace \tilde{V}_{Ψ^0} , which will be needed to check the third condition in theorem 1. It is the subspace of all vectors Ψ , for which $[\mathfrak{A}(\Psi)]_{22} = 0$. Observe that $\Psi^- \in \tilde{V}_{\Psi^0}$, which is in contradiction with the third condition in theorem 1. No nonzero vector thus can be in the kernel. \square

Because we know that any EW in $\mathbb{C}^2 \otimes \mathbb{C}^2$ is decomposable [12], we conclude that the partial transposition of an entangled positive matrix in $\mathbb{C}^2 \otimes \mathbb{C}^2$ has exactly one negative and three positive eigenvalues. For another proof of this fact, see [13].

4. Conditions on eigenvalues of entanglement witnesses

Theorem 2 assures, that given two semipositive observables W_{\pm} which have orthogonal supports, when some conditions on the support of W_- are fulfilled, one can construct a witness $\lambda W_+ - W_-$ for sufficiently large alpha. There is still a question of how big λ should be. We restrict ourselves in this subsection to entanglement witnesses, i.e. we fix $k = 1$. In some cases, for $k = 1$ parameter λ can be calculated or at least it acquires some new interpretation.

In the beginning, take $W_- = P_{V_-}$ and $W_+ = I - P_{V_-}$. Because $V_0 = (V_+ \oplus V_-)^\perp = \{0\}$, the third condition of theorem 2 is fulfilled in a trivial way. The second one in this situation reads $V_- \cap S_k = \emptyset$. Let us take therefore a subspace V_- which does not contain any vectors of Schmidt rank one.

Up to normalization, the witness $\lambda W_+ - W_-$ is equal to

$$W = \epsilon I - P_{V_-}, \quad \epsilon = \frac{\lambda}{\lambda + 1}.$$

Because the observable is an entanglement witness, it has positive eigenvalues when restricted to separable states. For any normalized $\Psi \in \mathcal{S}_1$ we have therefore

$$\epsilon \geq \langle \Psi | P_{V_-} | \Psi \rangle = \|P_{V_-} \Psi\|^2 = (\sup\{|\langle \Phi | \Psi \rangle| : \Phi \in V_- \wedge \|\Phi\| = 1\})^2.$$

The smallest ϵ which fulfils the above condition is a supremum of the right-hand side with respect to all normalized $\Psi = \phi \otimes \chi$,

$$\epsilon_{\min} = (\sup\{|\langle \Phi | \phi \otimes \chi \rangle| : \|\phi\| = 1 \wedge \|\chi\| = 1 \wedge \|\Phi\| = 1 \wedge \Phi \in V_-\})^2.$$

It is easy to observe, that $\langle \Phi | \phi \otimes \chi \rangle = \phi^T \mathfrak{A}(\Phi) \chi$. Using it, we find the final formula for ϵ :

$$\begin{aligned} \epsilon_{\min} &= (\sup\{|\phi^T \mathfrak{A}(\Phi) \chi| : \|\phi\| = 1 \wedge \|\chi\| = 1 \wedge \|\Phi\| = 1 \wedge \Phi \in V_-\})^2 \\ &= (\sup\{\|\mathfrak{A}(\Phi)\| : \|\Phi\| = 1 \wedge \Phi \in V_-\})^2 \\ &= \sup\{\|\mathfrak{A}(\Phi)\|^2 : \|\Phi\| = 1 \wedge \Phi \in V_-\}. \end{aligned}$$

The supremum norm of $\mathfrak{A}(\Psi)$ is the maximal Schmidt coefficient of the vector Ψ . Translating by the isomorphism \mathfrak{A} the subspace V_- into the corresponding subspace $\mathfrak{A}(V_-)$ in the space of matrices of coefficients, we are looking for the supremum of the sup norm on the unit sphere in this subspace. The quantity ϵ_{\min} can be therefore interpreted as the *supremum norm of the subspace of matrices*. We will therefore denote it further as $\epsilon_{\min} = \|\mathfrak{A}(V_-)\|_{\text{sup}}^2$. When the subspace V_- is one-dimensional and spanned by a vector Ψ , ϵ_{\min} can be calculated as $\|\mathfrak{A}(\Psi)\|_{\text{sup}}^2$. We can use this special class of witnesses to find the conditions for the eigenvalues of entanglement witnesses. One can estimate any witness $W = W_+ - W_-$ from above as

$$\begin{aligned} W_+ - W_- &\leq \lambda_+^{\max} P_{V_+} - \lambda_-^{\min} P_{V_-} \leq \lambda_+^{\max} P_{V_+ \oplus V_0} - \lambda_-^{\min} P_{V_-} \\ &= \lambda_+^{\max} I - (\lambda_+^{\max} + \lambda_-^{\min}) P_{V_-}, \end{aligned} \tag{14}$$

and from below as

$$\begin{aligned} W_+ - W_- &\geq \lambda_{\min}^+ P_{V_+} - \lambda_{\max}^- P_{V_-} \geq \lambda_+^{\min} P_{V_+} - \lambda_-^{\max} P_{V_- \oplus V_0} \\ &= \lambda_+^{\min} I - (\lambda_+^{\min} + \lambda_-^{\max}) P_{V_- \oplus V_0}. \end{aligned} \tag{15}$$

The upper bound is EW iff

$$\frac{\lambda_+^{\max}}{\lambda_+^{\max} + \lambda_-^{\min}} \leq \|\mathfrak{A}(V_-)\|_{\text{sup}}^2. \tag{16}$$

We conclude therefore, that (16) is a necessary condition for W to be an entanglement witness.

The lower bound is EW iff

$$\frac{\lambda_+^{\min}}{\lambda_+^{\min} + \lambda_-^{\max}} \leq \|\mathfrak{A}(V_- \oplus V_0)\|_{\text{sup}}^2. \tag{17}$$

We conclude now, that (17) is a sufficient condition for W to be an entanglement witness. Observe, that this sufficient condition can be fulfilled iff there are no vectors of Schmidt rank 1 in the kernel.

When the negative subspace of W is one-dimensional and the kernel is trivial, then one gets the sufficient condition

$$\frac{\lambda_+^{\min}}{\lambda_+^{\min} + \lambda_-} \leq \|\mathfrak{A}(\Psi)\|_{\text{sup}}^2.$$

One can find this condition for related positive maps in [14].

It rests to show how theorem 2 works when there exist vectors of Schmidt rank 1 in the kernel. In this example, consider the space $\mathbb{C}^3 \otimes \mathbb{C}^4$. Denote by $\{e_i\}_{i=1}^3$ the basis of \mathbb{C}^3 and by $\{f_i\}_{i=1}^4$ the basis of \mathbb{C}^4 . Let us span the kernel by two vectors: $e_1 \otimes f_1$ and $e_1 \otimes f_2$. The space \hat{V} is now

$$\hat{V} = \text{span}\{e_2, e_3\} \otimes \text{span}\{f_3, f_4\}.$$

The subspace V_- can be now embedded in a two-qubit space. It is therefore spanned by one normalized vector of Schmidt rank two. Denote it by ψ_- . Now we choose the arbitrary orthonormal basis of $V_-^\perp \cap \text{span}\{e_2, e_3\} \otimes \text{span}\{f_3, f_4\}$. Denote its elements by k_1, k_3 and k_2 . Finally, we complete the set $\{e_1 \otimes f_1, e_1 \otimes f_2, \psi_-, k_1, k_2, k_3\}$ to the orthonormal basis of $\mathbb{C}^3 \otimes \mathbb{C}^4$ by six vectors l_1, \dots, l_6 . It will be the basis of our witness.

Now theorem 2 assures, that fixing the negative eigenvalue, say -1 , and choosing large enough positive eigenvalues, one can get an entanglement witness. To do it, let us fix the eigenvalues related to the vectors k_1, k_3 and k_2 to be equal to $\epsilon/(1-\epsilon)$, where $\epsilon = \|\mathfrak{A}(\psi_-)\|_{\text{sup}}^2$. Suppose, that the rest of the eigenvalues are equal to zero—we have two-qubit witness whose domain is embedded in $\mathbb{C}^3 \otimes \mathbb{C}^4$. We can leave the eigenvalues related to the eigenvectors l_1, \dots, l_6 unchanged, or increase them to some positive values—it only destroys the optimality of the witness.

5. Translation to some properties of positivity-preserving mappings between the matrix algebras

The set of linear maps between the matrix algebras $\mathcal{B}(\mathbb{C}^{d_1})$ and $(\mathcal{B}(\mathbb{C}^{d_2}))$ and the set of bilinear forms on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ is isomorphic by the well-known Jamiołkowski isomorphisms [15] $J : \mathcal{B}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}) \rightarrow \mathcal{L}(\mathcal{B}(\mathbb{C}^{d_1}), \mathcal{B}(\mathbb{C}^{d_2}))$,

$$W_\Lambda = J(\Lambda) = [I \otimes \Lambda]|\Psi^+\rangle\langle\Psi^+|, \tag{18}$$

where $\mathfrak{A}(\Psi^+) = 1/d_1 \text{Id}_{d_1}$ (a projector onto Ψ^+ is the maximally entangled state).

The subset of linear maps, which preserve the Hermiticity of a matrix, is isomorphic by (18) to the subset of Hermitian bilinear forms. Any Hermitian matrix has spectral decomposition

$$W = \sum_{i=1}^p \lambda_i^+ |\Psi_i^+\rangle\langle\Psi_i^+| - \sum_{i=1}^q |\lambda_i^-| |\Psi_i^-\rangle\langle\Psi_i^-|, \tag{19}$$

where (p, q) is the signature of W . Such a matrix is related by the isomorphism (18) to a Hermiticity-preserving linear map:

$$\Lambda\rho = \sum_{i=1}^p A_i \rho A_i^\dagger - \sum_{i=1}^q B_i \rho B_i^\dagger, \tag{20}$$

where $A_i = \sqrt{\lambda_i^+} \mathfrak{A}(\Psi_i^+)$ and $B_i = \sqrt{|\lambda_i^-|} \mathfrak{A}(\Psi_i^-)$. For more details and other facts about this correspondence see [16] and the references therein.

The matrices A_i and B_i are orthogonal with respect to the Hilbert–Schmidt inner product. The observable W admits many decompositions into linear combinations of projectors, but the decomposition into a combination of orthogonal projectors is unique up to degeneracy of eigenvalues. Similarly, one Hermiticity-preserving map Λ admits many decompositions into the form (20) (such a form is called the *Kraus–Choi form* of Hermiticity preserving map due to the Kraus–Choi representation theorem [17]), but if we assume the orthogonality of matrices A_i and B_i , then the decomposition becomes unique.

In the set of maps, which preserve the Hermiticity we are allowed to define a subset of maps which preserve positivity. Such maps are called *positive maps*. We can generalize the definition of a positive map to a k -positive map such that Λ is k -positive when the map $Id_k \otimes \Lambda$ is positive. The isomorphism (18) relates a k -positive map to a k -Schmidt witness. The isomorphism (18) allows us to reformulate the propositions about properties of k -SWs to propositions describing properties of k -positive maps. Observe that if matrices A_i, B_i in (20) are not orthogonal, but still linearly independent, then (p, q) remains unchanged.

Consider now a Hermiticity-preserving map in its Kraus–Choi form:

$$\Lambda\rho = \sum_{i=1}^p A_i\rho A_i^\dagger - \sum_{i=1}^q B_i\rho B_i^\dagger. \quad (21)$$

Assume that matrices A_i, B_i are linearly independent. We have then the following:

Proposition 4. *If Λ is a k -positive map, then $0 \leq q \leq (d_1 - k) \times (d_2 - k)$.*

Proposition 5. *If Λ is a k -positive map and does not map any state of rank less or equal to k to zero, then $p \geq d_1 d_2 - (d_1 - k) \times (d_2 - k)$.*

6. Conclusions

The necessary and sufficient condition for a k -Schmidt witness has been presented and proved. Moreover, in the system of two qubits, the necessary condition fully determines the spectral type of any entanglement witness. The necessary condition also provides an upper bound for the dimension of the negative subspace of a k -Schmidt witness in arbitrary dimensions, which can be translated to analogous properties of k -positive maps. Also the sufficient condition for a k -Schmidt witness (up to rescaling of the positive part of an observable) has been proved.

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